

Free divisors and rational cuspidal curves

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References

1. *A. Dimca, G. Sticlaru, Free divisors and rational cuspidal plane curves*, arxiv:1504.01242
2. *A. Dimca, G. Sticlaru, Nearly free divisors and rational cuspidal curves*, arxiv:1505.00666
3. *A. Dimca, Freeness versus maximal global Tjurina number for plane curves*, arxiv:1508.04954

- 1 The main characters of our story
 - Rational cuspidal curves
 - Free divisors in \mathbb{P}^2
 - Nearly free divisors in \mathbb{P}^2
- 2 Main conjectures and main results
- 3 Ingredients in the proofs
 - Uli Walther's result
 - Proof in the even degree case
 - A Lefschetz type property for the graded module $N(f)$ due to Dorin Popescu and the speaker

Rational cuspidal curves: definition and first examples

Let $C : f = 0$ be a reduced curve in the complex projective plane \mathbb{P}^2 , defined by a degree d homogeneous polynomial f in the graded polynomial ring $S = \mathbb{C}[x, y, z]$.

We say that C is a **rational cuspidal curve** (r.c.c.) if C is homeomorphic to \mathbb{P}^1 , i.e. C is irreducible, any singularity of C is unibranch and

$$\mu(C) = \sum_{x \in C_{\text{sing}}} \mu(C, x) = (d-1)(d-2),$$

the **maximal possible value** for $\mu(C)$ if d is fixed. Indeed

$$g = \frac{(d-1)(d-2)}{2} - \sum \delta_x$$

with

$$\delta_x = \frac{\mu_x + r_x - 1}{2} = \frac{\mu_x}{2}$$

when $r_x = 1$. Here $\mu(C, x) = \mu_x$.

Example (E. Artal Bartolo, T. K. Moe, M. Namba)

- ① If $d = 1$, then C is a line, e.g. $x = 0$;
- ② If $d = 2$, then C is a smooth conic, e.g. $x^2 + y^2 + z^2 = 0$;
- ③ If $d = 3$, then C is a cuspidal cubic, e.g. $x^2y + z^3 = 0$;
- ④ If $d = 4$, then up to projective equivalence, there are 5 possibilities for C , having respectively 1, 2 or 3 cusps. For example,

$$3A_2 : x^2y^2 + y^2z^2 + x^2z^2 - 2xyz(x + y + z) = 0$$

$$A_2A_4 : z^4 - xz^3 - 2xyz^2 + x^2y^2 = 0, \quad A_6 : y^4 - 2xy^2z + yz^3 + x^2z^2 = 0.$$

- ⑤ If $d = 5$, then up to projective equivalence, there are 11 possibilities for C , having respectively 1, 2, 3 or 4 cusps. Any r.c.c. with 4 cusps is projectively equivalent to

$$3A_2A_6 : 16x^4y + 128x^2y^2z - 4x^3z^2 + 256y^3z^2 - 144xyz^3 + 27z^5 = 0.$$

Classification results I

A r.c.c. C is **unicuspidal** if it has a unique singularity. Here is the (topological) classification of such curves having a **unique Puiseux pair**.

Unique Puiseux pair (a, b) means same topology as the cusp $u^a + v^b = 0$, in particular $g.c.d.(a, b) = 1$. However, the analytic type, e.g. expressed by the Tjurina number

$$\tau(g) = \dim \mathcal{O}_2 / (g_u, g_v, g) \leq \mu(g) = \dim \mathcal{O}_2 / (g_u, g_v) = (a-1)(b-1)$$

can vary. The equality $\tau(g) = \mu(g)$ holds iff $g = 0$ is a weighted homogeneous singularity. The **global Tjurina number** is

$\tau(C) = \sum_{x \in C_{\text{sing}}} \tau(C, x)$, and one may ask about the maximal possible value of $\tau(C)$.

Theorem (J. Fernandez de Bobadilla, I. Luengo, A. Melle-Hernandez, A. Némethi, 2004)

Let a_j be the Fibonacci numbers with $a_0 = 0$, $a_1 = 1$, $a_{j+2} = a_{j+1} + a_j$. A Puiseux pair (a, b) can be realized by a unicuspidal rational curve of degree $d \geq 3$ if and only if the triple (a, b, d) occurs in the following list.

- 1 $(d - 1, d, d)$;
- 2 $(d/2, 2d - 1, d)$ with d even;
- 3 $(a_{j-2}^2, a_j^2, a_{j-2}a_j)$ with $j \geq 5$ odd;
- 4 (a_{j-2}, a_{j+2}, a_j) with $j \geq 5$ odd;
- 5 $(3, 22, 8)$ and $(6, 43, 16)$.

Classification results II

We say a r.c.c. C has **type** (d, m) if d is the degree of C and m is the highest multiplicity of its singularities. The classification of curves of type $(d, d - 1)$ is rather easy, and the curves of type $(d, d - 2)$ have been classified by Flenner-Zaidenberg (1996), Sakai-Tono (2000). The largest cusp can have the first Puiseux pair either $(d - 2, d)$ (case (i) in the next result) or $(d - 2, d - 1)$ (case (ii)).

Here is a **sample** of this classification. Let C be a r.c.c. of type $(d, d - 2)$ having **two cusps**, let's say q_1 of multiplicity $d - 2$ and q_2 of multiplicity $\leq d - 2$. Then the following cases are possible.

Proposition

(i) The germ (C, q_2) is a singularity of type A_{2d-4} . Then for each $d \geq 4$, C is unique up to projective equivalence.

(ii) The germ (C, q_2) is a singularity of type A_{d-1} , with $d = 2k + 1 \geq 5$ odd. Up to projective equivalence the equation of C can be written as

$$C : f = (y^{k-1}z + \sum_{i=2,k} a_i x^i y^{k-i})^2 y - x^{2k+1} = 0.$$

(iii) The germ (C_d, q_2) is a singularity of type A_{2j} , with d even and $1 \leq j \leq (d-2)/2$. Up to projective equivalence the equation of C_d can be written as

$$C : f = (y^{k+j}z + \sum_{i=2,k+j+1} a_i x^i y^{k+j+1-i})^2 - x^{2j+1} y^{2k+1} = 0,$$

where $a_{k+j+1} \neq 0$, $d = 2k + 2j + 2 \geq 6$, $k \geq 0$, $j \geq 1$.

Conclusions

1. Classification of r.c.c. is very difficult and complex, maybe impossible.
2. However, the r.c.c. are very interesting objects of study. The **Coolidge-Nagata conjecture** (any r.c.c. can be transformed into a line using some birational morphisms of \mathbb{P}^2) was finally proved in 2015 by Mariusz Koras and Karol Palka.

And there are a number of interesting open questions, as the following one.

Question

What is the maximal number N of cusps a r.c.c. can have?

Known examples suggest $N = 4$, K. Tono has shown $N \leq 8$ (2005), K. Palka improved this bound to $N \leq 6$ (2014).

Free divisors in \mathbb{P}^2

Let J_f be the **Jacobian ideal** of f , i.e. the homogeneous ideal of S spanned by the partial derivatives f_x, f_y, f_z of f and let $M(f) = S/J_f$ be the corresponding graded ring, called the **Jacobian** (or **Milnor**) **algebra** of f .

Let I_f denote the **saturation** of the ideal J_f with respect to the maximal ideal $\mathfrak{m} = (x, y, z)$ in S , i.e.

$$I_f = \{h \in S : \mathfrak{m}^k h \subset J_f \text{ for some integer } k > 0\}.$$

It is known that $I_f/J_f = H_{\mathfrak{m}}^0(M(f))$ is a finite dimensional \mathbb{C} -vector space.

Freeness in the local analytic setting was introduced by K. Saito (1980). Here we consider the graded version of this notion.

Definition

The curve $C : f = 0$ is a **free divisor** if the following two equivalent conditions hold.

- 1 $N(f) := I_f/J_f = 0$, i.e. the Jacobian ideal is saturated.
- 2 The minimal resolution of the Milnor algebra $M(f)$ has the following (short) form

$$0 \rightarrow S(-d_1 - d + 1) \oplus S(-d_2 - d + 1) \rightarrow S^3(-d + 1) \rightarrow S.$$

When C is a free divisor, the integers $d_1 \leq d_2$ are called the **exponents** of C . They satisfy the relations

$$d_1 + d_2 = d - 1 \text{ and } d_1 d_2 = (d - 1)^2 - \tau(C),$$

where $\tau(C)$ is the total Tjurina number of C .

First properties of free divisors

Let $m(f)_k = \dim M(f)_k$ and $n(f)_k = \dim N(f)_k$. It is known that $N(f)$ is **self-dual**, in particular $n(f)_k = n(f)_{T-k}$ for any k , where $T = 3(d-2)$. One can also show that

$$(\star) \quad n(f)_k = m(f)_k + m(f)_{T-k} - m(f_s)_k - \tau(C),$$

where f_s is a homogeneous polynomial of degree d with $C_s : f_s = 0$ smooth, e.g. $f_s = x^d + y^d + z^d$.

Corollary

The freeness of $C : f = 0$ is determined by the sequence $m(f)_k$. Conversely, if $C : f = 0$ is free, the sequence $m(f)_k$ is determined by the exponents d_1 and d_2 .

The sequence $m(f)_k$ is just the Hilbert function of the graded Milnor algebra $M(f)$.

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A result of A. A. du Plessis and C.T.C. Wall (1999)

The minimal degree of a Jacobian relation for f is the integer $mdr(f)$ defined to be the smallest integer $m \geq 0$ such that there is a nontrivial relation

$$af_x + bf_y + cf_z = 0 \quad (1)$$

among the partial derivatives f_x, f_y and f_z of f with coefficients a, b, c in S_m , the vector space of homogeneous polynomials of degree m .

Theorem

If $r = mdr(f) < d/2$, then one has

$$\tau(C) \leq \tau(d, r)_{\max} := (d-1)(d-r-1) + r^2,$$

and equality holds if and only if $C : f = 0$ is a free curve.

Remark: $\tau(C)$ is the Hilbert polynomial of $M(f)$.

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Irreducible free divisors

In this talk we concentrate mainly on **irreducible** free divisors in \mathbb{P}^2 . Only few examples of **irreducible** free divisors were known until recently, see A. Simis and S. Tohăneanu (2011) and R. Nanduri (2013).

Example

The example given by A. Simis and S. Tohăneanu

$$C_d : f_d = y^{d-1}z + x^d + ax^2y^{d-2} + bxy^{d-1} + cy^d = 0, \quad a \neq 0$$

is a family of free rational cuspidal curves for $d \geq 5$.

This example has the following properties:

- (i) the rationality of C_d is obvious, since $z = R(x, y)$.
- (ii) $\tau(C_d) < \mu(C_d)$ for $d > 5$, i.e. C_d has some non w.h. singularity.

We can show following.

Theorem

Any r.c.c. C which is free with exponents d_1, d_2 satisfies

$$d = d_1 + d_2 + 1 \geq 5, \quad d_1 \geq 2 \text{ and } \mu(C) - \tau(C) = (d_1 - 1)(d_2 - 1) - 1.$$

In particular $\mu(C) > \tau(C)$ unless $d = 5$, when one has $d_1 = d_2 = 2$.

Theorem (First experimental meta theorem)

About half of the rational cuspidal curves which occur in the various classification lists are free divisors.

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Nearly free divisors: definition and first properties

What about the other half?

Definition

The curve $C : f = 0$ is a **nearly free divisor** if the following two equivalent conditions hold.

- 1 $N(f) \neq 0$ and $n(f)_k \leq 1$ for any k .
- 2 the Milnor algebra $M(f)$ has a minimal resolution of the form

$$0 \rightarrow S(-d-d_2) \rightarrow S(-d-d_1+1) \oplus S^2(-d-d_2+1) \rightarrow S^3(-d+1) \rightarrow S$$

Theorem

If $C : f = 0$ is nearly free, then the exponents $d_1 \leq d_2$ satisfy $d_1 + d_2 = d$ and $\tau(C) = (d-1)^2 - d_1(d_2-1) - 1$. In particular, there are 3 syzygies R_1, R_2, R_3 of degrees d_1, d_2, d_2 satisfying one second order syzygy $hR_1 + \ell_2R_2 + \ell_3R_3 = 0$, with degree of h, ℓ_2, ℓ_3 equal to $d_2 - d_1 + 1, 1, 1$ and ℓ_2, ℓ_3 linearly independent.

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We can show following.

Theorem

If $r = \text{mdr}(f) \leq d/2$, then one has

$$\tau(C) = \tau(d, r)_{\max} - 1$$

if and only if C is a nearly free curve.

The case $r = 1$ deserves special attention.

Corollary

(i) One has $\text{mdr}(f) = 1$ if and only if C admits a 1-dimensional symmetry, i.e. C admits a 1-dimensional algebraic subgroup of $\text{PGL}_2(\mathbb{C})$ as automorphism group.

(ii) If $\text{mdr}(f) = 1$, then C is either free or nearly free. If in addition C is irreducible, then C is nearly free.

Theorem (Second experimental meta theorem)

All the rational cuspidal curves which occur in the various classification lists are either free or nearly divisors.

Example

The curve $C_d : f_d = x^d + y^{d-1}z = 0$ is nearly free for $d \geq 2$ with exponents $(d_1, d_2) = (1, d-1)$. In addition $n(f)_k = 1$ for $d-2 \leq k \leq 2d-4$ and $n(f)_k = 0$ otherwise.

Conjecture (First Main Conjecture)

Any rational cuspidal curve is either free or nearly free.

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Theorem (Main result until recently!)

Let $C : f = 0$ be a rational cuspidal curve of degree d . Assume that either

- 1 d is even, or
- 2 d is odd and for any singularity x of C , the order of any eigenvalue λ_x of the local monodromy operator h_x is not d .

Then C is either a free or a nearly free curve.

Corollary (until recently!)

Let $C : f = 0$ be a rational cuspidal curve of degree d such that

- 1 either $d = p^k$ is a prime power, or
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Conjecture (Second Main Conjecture)

An irreducible plane curve which either free or nearly free is rational.

Remark

Non-linear free arrangements can have non-rational irreducible components. Here is one example due to Jean Vallès.

$$xyz(x^3 + y^3 + z^3)[(x^3 + y^3 + z^3)^3 - 27x^3y^3z^3] = 0.$$

Let $\nu : \tilde{C} \rightarrow C$ be the normalization of the irreducible curve C . Assume that the singularities of C are the points p_i , $i = 1, \dots, q$ and the germ (C, p_i) has r_i branches.

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Theorem

(i) If C is an irreducible free curve of degree d , then $d \geq 5$ and

$$2g(\tilde{C}) + \sum_{i=1,q} (r_i - 1) \leq \frac{(d-1)(d-5)}{4}.$$

In particular, if $d \leq 6$, then C is rational and $r_i \leq 2$ for all i , with equality for at most one i .

(ii) If C is an irreducible nearly free curve of degree d such that $d_1 = 1$ (resp. $d \leq 5$), then C is rational cuspidal (resp. rational and $r_i \leq 2$ for all i , with equality for at most one i).

There are examples of free (resp. nearly free) rational curves C having one singularity p_i with $r_i = 2$ (resp. $r_i = 3$).

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Relation to the monodromy

The preprint [U. Walther, The Jacobian module, the Milnor fiber, and the \$D\$ -module generated by \$f^s\$, arXiv:1504.07164](#) states that one has an **injection**

$$N(f)_{2d-2-j} \rightarrow H^2(F, \mathbb{C})_\lambda,$$

with $j = 1, 2, \dots, d$, where $F : f(x, y, z) - 1 = 0$ is the **Milnor fiber** associated to C and the subscript λ indicates the **eigenspace of the monodromy action** corresponding to the eigenvalue $\lambda = \exp(2\pi i(d + 1 - j)/d)$. But recently U. Walther found an error in his proof.

The preprint [Morihiro Saito, Hilbert series of graded Milnor algebras and roots of Bernstein-Sato polynomials, arXiv:1509.06288](#) gives an example where the morphism supposed to give the above injection is not an injection **after passing to the corresponding graded morphism**. However, even in this case one has $\dim N(f)_{2d-2-j} \leq \dim H^2(F, \mathbb{C})_\lambda$, and this is all we need for our proofs.

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Proof in the case d even

Suppose now that d is even, say $d = 2d_1$. **It is enough to show that** $\dim H^2(F, \mathbb{C})_\lambda = 1$ for $\lambda = -1$ which corresponds to $j = d_1 + 1$.

Note first that -1 is not an eigenvalue for the local monodromy h_x of any cusp (C, x) : knot theory implies that $h_x - Id$ is a presentation matrix of the \mathbb{Z} -module $H_1(S^3 \setminus L_x) = \mathbb{Z}$.

The fact that $\lambda = -1$ is not an eigenvalue for the local monodromy h_x of any cusp (C, x) implies that $H^1(F)_\lambda = 0$.

Since $E(U) = E(\mathbb{P}^2) - E(C) = 1$, it follows that

$$\dim H^2(F, \mathbb{C})_\lambda - \dim H^1(F, \mathbb{C})_\lambda + \dim H^0(F, \mathbb{C})_\lambda = 1.$$

Since clearly $H^0(F)_\lambda = 0$, the result is proved.

A Lefschetz type property for $N(f)$

The preprint

A. Dimca, D. Popescu, Hilbert series and Lefschetz properties of dimension one almost complete intersections, arXiv:1403.5921.

implies the following result, used **twice** in the proofs (case d even of the main theorem and the form of the minimal resolution of a nearly free divisor). Idea of proof: splitting of the syzygy bundle restricted to lines as in H. Brenner, A. Kaid (2007).

Theorem

If $C : f = 0$ is a degree d reduced plane curve, then there exists a Lefschetz element for $N(f)$. More precisely, for a generic linear form $\ell \in S_1$, the multiplication by ℓ induces injective morphisms $N(f)_i \rightarrow N(f)_{i+1}$ for integers $i < T/2$ and surjective morphisms $N(f)_i \rightarrow N(f)_{i+1}$ for integers $i \geq i_0 = \lceil T/2 \rceil$. In particular one has

$$0 \leq n(f)_0 \leq n(f)_1 \leq \dots \leq n(f)_{\lceil T/2 \rceil} \geq n(f)_{\lceil T/2 \rceil + 1} \geq \dots \geq n(f)_T \geq 0.$$

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